DISSERTATIO MATHEMATICA
DE
TRACTORIIS SECTIONUM CONICARUM

CUJUS PARTEM SECUNDAM

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PRÆSIDE

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ABOÆ,
Typis FRENCKELLIANIS.

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Schol. Æquatio quam pro Tractoria ſimplici jam invenimus, ope methodi tangentium directæ facillime inveſtigari poteſt. Valor etenim tangentis generalis $y\,\frac{\sqrt{dx^2+dy^2}}{dy}$ æqualis eſt ponendus quantitati b, unde $b = y\,\frac{\sqrt{dx^2+dy^2}}{dy}$ & facta debita reductione, $dx = dy\,\frac{\sqrt{b^2-y^2}}{y}$, eadem nempe quam ſupra invenimus.

§. 4.

Exiſtente linea ATL Circulo, radio r deſcripto, cujus æquatio ſit $u^2 = 2r\xi - \xi^2$, ponatur tangens $MT = b$, habebitur comparatis æquationibus (A) & (B), $ds = \frac{brd\phi}{(r\phi-b)\sqrt{1-\phi^2}} = \frac{dz}{\sqrt{1-\phi^2}}$; adeoque $dz = \frac{brd\phi}{r\phi-b}$ & peracta integratione $z = b\,Log\,(r\phi-b)$, unde ſi N fuerit numerus cujus logarithmus Hyperbolicus $= 1$, erit $N^{\frac{z}{b}} = r\phi - b. + C.$

Ex æquatione autem (C) eruitur $u = \frac{ydz - bdy}{dz}$.

& hinc $du = \frac{bdyddz - bdzddy + dydz^2}{dz^2}$, atque

$u^2 = \frac{(ydz - bdy)^2}{dz^2}$. Est autem $ds = \sqrt{du^2 + d\xi^2}$, &

$ds^2 = du^2 + d\xi^2 = du^2 + \frac{u^2 du^2}{r^2 - u^2} = \frac{r^2 du^2}{r^2 - u^2}$, unde itaque

$ds = \frac{r du}{\sqrt{r^2 - u^2}}$, & $ds\sqrt{1 - \varphi^2} = dz = \frac{rdu\sqrt{1 - \varphi^2}}{\sqrt{r^2 - u^2}}$; adeoque

$r^2 - u^2 = \frac{r^2 du^2 (1 - \varphi^2)}{dz^2}$. Erat autem $u^2 = \frac{(ydz - bdy)^2}{dz^2}$,

ergo $r^2 - u^2 = r^2 - \frac{(ydz - bdy)^2}{dz^2}$ & $r^2 du^2 (1 - \varphi^2) =$

$r^2 dz^2 - (ydz - bdy)^2$; quum vero sit $r\varphi - b = N\frac{z}{b}$

erit $\varphi = \frac{N\frac{z}{b} + b}{r}$, atque hinc

$r^2 dz^2 - (ydz - bdy)^2 = \frac{(bdyddz - bdzddy + dydz^2)^2}{dz^4} \times$

$(r^2 - (N\frac{z}{b} + b)^2)$ substituto loco du valore ejus jam invento. In hac quidem æquatione non occurrunt nisi functiones coordinatarum x & y, ipsa vero æquatio ad curvam pendet ex integratione inventæ jam æqua-

æquationis, quæ secundum regulas nobis hucusque cognitas integrari non potest.

Ad Tractoriam vero Circuli construendam, sequens nobis satis commoda videtur methodus: ex æquatione (A) §. 2. eruitur

$ds = \frac{brd\varphi}{(r\varphi - b)\sqrt{1-\varphi^2}} = \frac{-brd\varphi}{(b-r\varphi)\sqrt{1-\varphi^2}}$; integrale hujus æquationis, quo facilius innotescat, statuatur

$\frac{b(1-\sqrt{1-\varphi^2})}{\varphi} = p$, sive $\varphi = \frac{2bp}{b^2+p^2}$, unde

$d\varphi = \frac{2bdp\,(b^2-p^2)}{(b^2+p^2)^2}$, atque $-\frac{brd\varphi}{(b-r\varphi)\sqrt{1-\varphi 2}}$

$= -\frac{2br\,dp}{b^2-2rp+p^2}$, cujus formulæ integrale pro diversis ipsarum r & b valoribus, vel absolutum evadit, vel a Logarithmis vel a rectificatione Arcus circularis pendet. Quum enim denominator $b^2 - 2rp + p^2$ binos factores simplices contineat, sive reales, eosdemque vel æquales vel inæquales, sive imaginarios, prout scilicet fuerit vel $r > b$ vel $r = b$ vel denique $r < b$, pro diversis his casibus integrale æquationis

$ds = -\frac{2brd\varphi}{b^2 - 2rp + p^2}$ eruendum erit.

Si itaque fuerit $r > b$ habebitur $s = \int \frac{-2brdp}{b^2 - 2rp + p^2}$.

 =

$$= \int \frac{br}{\sqrt{r^2 - b^2}} \left(\frac{-dp}{p - r - \sqrt{r^2 - b^2}} + \frac{dp}{p - r + \sqrt{r^2 - b^2}} \right)$$

$$= \frac{br}{\sqrt{r^2 - b^2}} \times (Log\ (p - r + \sqrt{r^2 - b^2}) -$$

$$Log\ (p - r - \sqrt{r^2 - b^2})) + C = \frac{br}{\sqrt{r^2 - b^2}}\ Log$$

$$\left(\frac{p - r + \sqrt{r^2 - b^2}}{p - r - \sqrt{r^2 - b^2}} \right) + C = \frac{br}{\sqrt{r^2 - b^2}}\ Log$$

$\dfrac{b(1 - \sqrt{1 - \varphi^2}) - \varphi(r - \sqrt{r^2 - b^2})}{b(1 - \sqrt{1 - \varphi^2}) - \varphi(r + \sqrt{r^2 - b^2})} + C$, restituto valore ipsius $p = \dfrac{b}{\varphi}(1 - \sqrt{1 - \varphi^2})$. In casu vero quo $r = b$, æquatio nostra in hanc abit formam: $ds = \dfrac{2r^2 dp}{(r - p)^2}$ facta eadem substitutione quam supra adhibuimus; unde peracta integratione eruitur $s = -\dfrac{2r^2}{r - p} + C = \dfrac{2r^2}{p - r} + C$; & si loco p adhibetur valor ejus supra assumtus erit $s = \dfrac{2r\varphi}{1 - \varphi - \sqrt{1 - \varphi^2}} + C$.

Si denique fuerit $r < b$, æquatio allata $ds = -\dfrac{2brdp}{b^2 - 2rp + p^2}$ ita transformari potest, ut fiat $ds = -\dfrac{2brdp}{b^2 - r^2 + (r - p)^2}$ & si integretur

$s =$

$s = -\frac{2br}{\sqrt{b^2 - r^2}}$ Arc. Tang $\frac{p - r}{\sqrt{b^2 - r^2}} + C$, atque restituto valore ipsius p, habebitur $s = -\frac{2br}{\sqrt{b^2 - r^2}} \times$ *Arc. Tang.* $\frac{b(1 - \sqrt{1 - \varphi^2}) - r\varphi}{\varphi\sqrt{b^2 - r^2}} + C.$ Constructio itaque Tractoriae in quovis casu innotescit.

Rectificatio hujus curvae ex supra allatis facillime determinatur; erat enim elementum Arcus $dz = \frac{brd\varphi}{r\varphi - b}$, adeoque $z = b\ Log\ (r\varphi - b) + C.$ Aream vero Tractoriae circuli $= A$ ita determinamus, ut elementum ipsius dA aequale assumamus differentiae triangulorum CtT & MTt; erit itaque $dA = \triangle CTt - \triangle MTt = \frac{CT.Tt}{2} - \frac{MT.tk}{2} = \frac{rds - b\varphi ds}{2}$ (§. 2), & si ex aequatione (A) depromatur valor ipsius ds, erit $dA = \frac{-br^2 d\varphi + b^2 r\varphi d\varphi}{2(b - r\varphi)\sqrt{1 - \varphi^2}} = \frac{br\ (b\varphi - r)d\varphi}{2\cdot(b - r\varphi)\sqrt{1 - \varphi^2}}$, quae aequatio facta $\varphi r - b = v$, hanc induit formam: $dA = \frac{b(r^2 - b^2)}{2} \times \frac{dv}{v\sqrt{r^2 - (b + v)^2}}$

$-b^2$

$-\frac{b^2}{2}.\frac{dv}{\sqrt{r^2-(b+v)^2}}$, & ſtatuendo $b+v=\frac{r\sqrt{p^2-1}}{p}$ habebitur $dA = \frac{dp}{\sqrt{p^2-1}.(r\sqrt{p^2-1}-bp)}.\frac{b(r^2-b^2)}{2}$ $-\frac{b^2}{2}.\frac{dp}{p\sqrt{p^2-1}}$, & integrando $A=-\frac{b^2}{2}$ *Arc. ſec.* p $+\int\frac{b(r^2-b^2)}{2}.\frac{dp}{\sqrt{p^2-1}.(r\sqrt{p^2-1}-bp)}$.

Ad inveniendum integrale membri poſterioris ponatur $p+\sqrt{p^2-1}=q$ ſeu $p=\frac{q^2+1}{2q}$, unde $\int\frac{b(r^2-b^2)}{2}.\frac{dp}{\sqrt{p^2-1}(r\sqrt{p^2-1}-bp)}=\int\frac{b(r^2-b^2)}{2}$ $\frac{2dq}{(r-b)q^2-r-b}$ facta debita terminorum reductione. Hæc autem formula reduci poteſt ad aliam huic æqualem $\int\frac{b(r+b)}{2}:\frac{2dq}{q^2-\frac{r+b}{r-b}}$, & poſito brevitatis cauſa $\frac{r+b}{r-b}=a^2$ atque multiplicato numeratore pariter ac denominatore per $-a$, prodit $\int\frac{b.\overline{r+b}}{-2a}.-\frac{2adq}{q^2-a^2}$, cujus integrale ſecundum regugulas cognitas est $\frac{b.\overline{r+b}}{-2a}.\,Log\,\frac{q+a}{q-a}=-\frac{b(r+b)}{2\sqrt{\frac{r+b}{r-b}}}\times$

$$log\,\frac{q+\sqrt{\frac{r+b}{r-b}}}{q-\sqrt{\frac{r+b}{r-b}}}=-\frac{b\sqrt{r^2-b^2}}{2}\,Log\,\frac{q+\sqrt{(r+b):(r-b)}}{q-\sqrt{(r+b):(r-b)}}$$

& restitutis valoribus quantitatum p & q, eruitur

$$A=-\frac{b\sqrt{r^2-b^2}}{2}\,Log\,\frac{(r+b+v)\sqrt{r-b}+\sqrt{r+b}\sqrt{r^2-(b+v)^2}}{(r+b+v)\sqrt{r-b}-\sqrt{r+b}\sqrt{r^2-(b+v)^2}}$$

$$-\frac{b^2}{2}\,Arc.\,Sec.\,\frac{r}{\sqrt{r^2-(b+v)^2}}+C.$$

Si sumto angulo NTM constante, ubique fiat $b=r\varphi$, Tractoriam hac ratione oriundam in Circulum abire concentricum, radio $\sqrt{r^2-b^2}$ describendum, perspicuum est. Junctis etenim punctis C & M, liquet fore angulum NTM rectum, unde $CM=\sqrt{CT^2-MT^2}=\sqrt{r^2-b^2}$. In casu vero quo angulus $NMT=90^\circ$, fiet $b=r$ & $\sqrt{r^2-b^2}=0$. Evanescente jam radio Circuli, ipsa Tractoria extra centrum Circuli non extenditur.

Paullo simplicior evadit æquatio, quam pro Tractoria Circuli supra invenimus, non adhibendis æquationibus (A) & (B) quantitatem φ involventibus;

bus; nam ob $\triangle Mom \sim \triangle MST$ erit mo ($= dy$): oM ($= dx$) :: MS ($= y - u$) : ST ($= x - \xi$), adeoque $dx(y - u) = dy(x - \xi)$; ex æquatione autem Circuli eruitur $\xi = r \pm \sqrt{r^2 - u^2}$, qui ipsius ξ valor, si substituatur in æquatione $dx(y - u) = dy(x - \xi)$, dabit $dx(y - u) = dy(x - r - \sqrt{r^2 - u^2})$. Sed æquatio (C) §. 2 exhibet $u = \frac{(ydz - bdy)}{dz}$, unde facta substitutione $bdx = (x - r).dz - \sqrt{r^2dz^2 - (ydz - bdy)^2}$ seu $(x - r).dz - bdx = \sqrt{(r^2dz^2 - (ydz - bdy)^2)}$. Atqui hinc $r^2dz^2 - y^2dz^2 + 2bydzdy - b^2dy^2 = x^2dz^2 - 2rxdz^2 + r^2dz^2 - 2b.(x - r)dxdz + b^2dx^2$, & facta debita reductione $dz(x^2 - 2rx + b^2 + y^2) = 2b(ydy + (x - r).dx) = \sqrt{dx^2 + dy^2} \times (x^2 - 2rx + b^2 + y^2)$.

§. 5.

Quod si fuerit ATL Parabola æquatione $u^2 = p\xi$ definita, ad æquationem pro Tractoria inveniendam, posito ut antea tangente MT constante æquali b, ex æquatione (B) desumitur $\sqrt{1 - \varphi^2} = \frac{dz}{ds}$ unde

$$\varphi = \frac{\sqrt{ds^2 - dz^2}}{ds} \text{ atque } d\varphi = \frac{dz^2 ds\, dds - ds\, dz\, ddz}{ds^2 \sqrt{ds^2 - dz^2}},$$

qui valores si in æquatione (A) substituantur, habebitur

$$\frac{dz^2 dds - ds\, dz\, ddz}{ds^2 \sqrt{ds^2 - dz^2}} = \frac{\sqrt{ds^2 - dz^2}}{b} - \frac{ds}{r}.$$